Introduction to Electromagnetism

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Gradient

In vector calculus, the gradient of a scalar field is a vector field that points in the direction of the greatest rate of increase of the scalar field, and whose magnitude is that rate of increase. In simple terms, the variation in space of any quantity can be represented (e.g. graphically) by a slope. The gradient represents the steepness and direction of that slope.

A generalization of the gradient for functions on a Euclidean space that have values in another Euclidean space is the Jacobian. A further generalization for a function from one Banach space to another is the Fréchet derivative.

Consider a room in which the temperature is given by a scalar field, *T*, so at each point (x,y,z) the temperature is $T(x,y,z)$. (We will assume that the temperature does not change over time.) At each point in the room, the gradient of *T* at that point will show the direction the temperature rises most quickly. The magnitude of the gradient will determine how fast the temperature rises in that direction.

Consider a surface whose height above sea level at a point (x, y) is $H(x, y)$. The gradient of H at a point is a vector pointing in the direction of the steepest slope or grade at that point. The steepness of the slope at that point is given by the magnitude of the gradient vector.

The gradient can also be used to measure how a scalar field changes in other directions, rather than just the direction of greatest change, by taking a dot product. Suppose that the steepest slope on a hill is 40%. If a road goes directly up the hill, then the steepest slope on the road will also be 40%. If, instead, the road goes around the hill at an angle, then it will have a shallower slope. For example, if the angle between the road and the uphill direction, projected onto the horizontal plane, is 60°, then the steepest slope along the road will be 20%, which is 40% times the cosine of 60°.

This observation can be mathematically stated as follows. If the hill height function H is differentiable, then the gradient of H dotted with a unit vector gives the slope of the hill in the direction of the vector. More precisely, when H is differentiable, the dot product of the gradient of H with a given unit vector is equal to the directional derivative of H in the direction of that unit vector.

Definition

The gradient (or gradient vector field) of a scalar function $f(x1, x2, x3, \ldots, xn)$ is denoted ∇f or where ∇ (the nabla symbol) denotes the vector differential operator, del. The notation "grad(f)" is also commonly used for the gradient. The gradient of f is defined as the unique vector field whose dot product with any vector **v** at each point x is the directional derivative of f along **v**. That is,

$$(\nabla f(x)) \cdot \mathbf{v} = D_{\mathbf{v}} f(x).$$

Divergence

In vector calculus, **divergence** is a vector operator that measures the magnitude of a vector field's source or sink at a given point, in terms of a signed scalar. More technically, the divergence represents the volume density of the outward flux of a vector field from an infinitesimal volume around a given point.

For example, consider air as it is heated or cooled. The relevant vector field for this example is the velocity of the moving air at a point. If air is heated in a region it will expand in all directions such that the velocity field points outward from that region. Therefore the divergence of the velocity field in that region would have a positive value, as the region is a source. If the air cools and contracts, the divergence is negative and the region is called a sink.

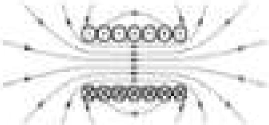
Definition of divergence

In physical terms, the divergence of a three dimensional vector field is the extent to which the vector field flow behaves like a source or a sink at a given point. It is a local measure of its "outgoingness"—the extent to which there is more exiting an infinitesimal region of space than entering it. If the divergence is nonzero at some point then there must be a source or sink at that position.(Note that we are imagining the vector field to be like the velocity vector field of a fluid (in motion) when we use the terms flow, sink and so on.)

Gauss's law

In physics, **Gauss's law**, also known as **Gauss's flux theorem**, is a law relating the distribution of electric charge to the resulting electric field.

The law was formulated by Carl Friedrich Gauss in 1835, but was not published until 1867.[1] It is one of the four Maxwell's equations which form the basis of classical electrodynamics, the other three being Gauss's law for magnetism, Faraday's law of induction, and Ampere's law with Maxwell's correction. Gauss's law can be used to derive Coulomb's law,[2] and vice versa.



Gauss's law can be stated using either the electric field **E** or the electric displacement field **D**. This section shows some of the forms with **E**; the form with **D** is below, as are other forms with **E**.

Integral form

Gauss's law may be expressed as:[3]

$$\Phi_E = \frac{Q}{\varepsilon_0}$$

where ΦE is the electric flux through a closed surface *S* enclosing any volume *V*, *Q* is the total charge enclosed within *S*, and $\varepsilon 0$ is the electric constant. The electric flux ΦE is defined as a surface integral of the electric field:

$$\Phi_E = \oiint_S \mathbf{E} \cdot \mathrm{d}\mathbf{A}$$

where **E** is the electric field, d**A** is a vector representing an infinitesimal element of area,[note 1] and · represents the dot product of two vectors.

Since the flux is defined as an integral of the electric field, this expression of Gauss's law is called the *integral form*.

If the electric field is known everywhere, Gauss's law makes it quite easy, in principle, to find the distribution of electric charge: The charge in any given region can be deduced by integrating the electric field to find the flux.

However, much more often, it is the reverse problem that needs to be solved: The electric charge distribution is known, and the electric field needs to be computed. This is much more difficult, since if you know the total flux through a given surface, that gives almost no information about the electric field, which (for all you know) could go in and out of the surface in arbitrarily complicated patterns.

An exception is if there is some symmetry in the situation, which mandates that the electric field passes through the surface in a uniform way. Then, if the total flux is known, the field itself can be deduced at every point. Common examples of symmetries which lend themselves to Gauss's law include cylindrical symmetry, planar symmetry, and spherical symmetry. See the article Gaussian surface for examples where these symmetries are exploited to compute electric fields.

Differential form

By the divergence theorem Gauss's law can alternatively be written in the *differential form*:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

where ∇·**E** is the divergence of the electric field, and ρ is the total electric charge density.

Electric displacement field

In physics, the **electric displacement field**, denoted by **D**, is a vector field that appears in Maxwell's equations. It accounts for the effects of free charge within materials. "D" stands for "displacement," as in the related concept of displacement current in dielectrics. In free space, the electric displacement field is equivalent to flux density, a concept that lends understanding to Gauss's law.

Definition

n a dielectric material the presence of an electric field **E** causes the bound charges in the material (atomic nuclei and their electrons) to slightly separate, inducing a local electric dipole moment. The electric displacement field **D** is defined as

$$\mathbf{D} \equiv \varepsilon_0 \mathbf{E} + \mathbf{P},$$

where

ε_0

is the vacuum permittivity (also called permittivity of free space), and **P** is the (macroscopic) density of the permanent and induced electric dipole moments in the material, called the polarization density. Separating the total volume charge density into free and bound charges:

$$\rho = \rho_f + \rho_b$$

the density can be rewritten as a function of the polarization **P**:

$$\rho = \rho_f - \nabla \cdot \mathbf{P}.$$

P is a vector field whose divergence yields the density of bound charges ρ_b in the material. The electric field satisfies the equation:

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}(\rho_f + \rho_b) = \frac{1}{\varepsilon_0}(\rho_f - \nabla \cdot \mathbf{P})$$

and hence

$$\nabla \cdot (\varepsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f$$

The displacement field therefore satisfies Gauss's law in a dielectric:

$$\nabla \cdot \mathbf{D} = \rho - \rho_b = \rho_f$$

D is not determined exclusively by the free charge. Consider the relationship:

$$\nabla \times \mathbf{D} = \varepsilon_0 \nabla \times \mathbf{E} + \nabla \times \mathbf{P}$$

Which, by the fact that **E** has a curl of zero in electrostatic situations, evaluates to:

$$\nabla \times \mathbf{D} = \nabla \times \mathbf{P}$$

Which can be immediately seen in the case of some object with a "frozen in" polarization like a bar electret, the electric analogue to a bar magnet. There is no free charge in such a material, but the inherent polarization gives rise to an electric field. If the wayward student were to assume the **D** field were entirely determined by the free charge, he or she would immediately conclude the electric field were zero in such a material, but this is patently not true. The electric field can be properly determined by using the above relation along with other boundary conditions on the polarization density yielding the bound charges, which will, in turn, yield the electric field.

In a linear, homogeneous, isotropic dielectric with instantaneous response to changes in the electric field, **P** depends linearly on the electric field,

$$\mathbf{P} = \varepsilon_0 \chi \mathbf{E},$$

where the constant of proportionality

χ

is called the electric susceptibility of the material. Thus

$$\mathbf{D} = \varepsilon_0 (1 + \chi) \mathbf{E} = \varepsilon \mathbf{E}$$

where $\varepsilon = \varepsilon_0 \varepsilon_r$ is the permittivity, and $\varepsilon_r = 1 + \chi$ the relative permittivity of the material.

In linear, homogeneous, isotropic media, ε is a constant. However, in linear anisotropic media it is a tensor, and in nonhomogeneous media it is a function of position inside the medium. It may also depend upon the electric field (nonlinear materials) and have a time dependent response. Explicit time dependence can arise if the materials are physically moving or changing in time (e.g. reflections off a moving interface give rise to Doppler shifts). A different form of time dependence can arise in a time-invariant medium, in that there can be a time delay between the imposition of the electric field and the resulting polarization of the material. In this case, **P** is a convolution of the impulse response susceptibility χ and the electric field **E**. Such a convolution takes on a simpler form in the frequency domain—by Fourier transforming the relationship and applying the convolution theorem, one obtains the following relation for a linear time-invariant medium:

$$\mathbf{D}(\omega) = \varepsilon(\omega)\mathbf{E}(\omega),$$

where

ω

is frequency of the applied field. The constraint of causality leads to the Kramers–Kronig relations, which place limitations upon the form of the frequency dependence. The phenomenon of a frequency-dependent permittivity is an example of material dispersion. In fact, all physical materials have some material dispersion because they cannot respond instantaneously to applied fields, but for many problems (those concerned with a narrow enough bandwidth) the frequency-dependence of ε can be neglected.

At a boundary,

$$(\mathbf{D_1} - \mathbf{D_2}) \cdot \hat{\mathbf{n}} = D_{1,\perp} - D_{2,\perp} = \sigma_f$$

, where σf is the free charge density and the unit normal

$\hat{\mathbf{n}}$

points in the direction from medium 2 to medium 1.[1]

Electromagnetic induction

Electromagnetic induction is the production of a potential difference (voltage) across a conductor when it is exposed to a varying magnetic field.

Michael Faraday is generally credited with the discovery of induction in 1831 though it may have been anticipated by the work of Francesco Zantedeschi in 1829.[1] Around 1830[2] to 1832,[3] Joseph Henry made a similar discovery, but did not publish his findings until later.

Faraday's law of induction is a basic law of electromagnetism that predicts how a magnetic field will interact with an electric circuit to produce an electromotive force (EMF). It is the fundamental operating principle of transformers, inductors, and many types of electrical motors, generators and solenoids. The **Maxwell–Faraday equation** is a generalization of Faraday's law, and forms one of Maxwell's equations.

Faraday's Law

Faraday's law of induction makes use of the magnetic flux Φ_B through a hypothetical surface Σ whose boundary is a wire loop. Since the wire loop may be moving, we write Σ(*t*) for the surface. The magnetic flux is defined by a surface integral:

$$\Phi_B = \iint_{\Sigma(t)} \mathbf{B}(\mathbf{r}, t) \cdot d\mathbf{A} \ ,$$

where dA is an element of surface area of the moving surface Σ(t), B is the magnetic field, and B·dA is a vector dot product (the infinitesimal amount of magnetic flux). In more visual terms, the magnetic flux through the wire loop is proportional to the number of magnetic flux lines that pass through the loop.

When the flux changes—because B changes, or because the wire loop is moved or deformed, or both—Faraday's law of induction says that the wire loop acquires an EMF

$$\mathcal{E}$$

, defined as the energy available per unit charge that travels once around the wire loop (the unit of EMF is the volt).[16][17][18][19] Equivalently, it is the voltage that would be measured by cutting the wire to create an open circuit, and attaching a voltmeter to the leads. According to the Lorentz force law (in SI units),

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

the EMF on a wire loop is:

$$\mathcal{E} = \frac{1}{q} \oint_{\mathrm{wire}} \mathbf{F} \cdot d\boldsymbol{\ell} = \oint_{\mathrm{wire}} \left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \cdot d\boldsymbol{\ell}$$

where E is the electric field, B is the magnetic field (aka magnetic flux density, magnetic induction), dℓ is an infinitesimal arc length along the wire, and the line integral is evaluated along the wire (along the curve the conincident with the shape of the wire).

The EMF is also given by the rate of change of the magnetic flux:

$$|\mathcal{E}| = \left|\frac{d\Phi_B}{dt}\right| \ ,$$

where

$$|\mathcal{E}|$$

is the magnitude of the electromotive force (EMF) in volts and ΦB is the magnetic flux in webers. The direction of the electromotive force is given by Lenz's law.

For a tightly wound coil of wire, composed of N identical loops, each with the same ΦB, Faraday's law of induction states that[20][21]

$$\mathcal{E} = N\frac{d\Phi_B}{dt}$$

where N is the number of turns of wire and ΦB is the magnetic flux in webers through a single loop.

Ampère's circuital law

In classical electromagnetism, **Ampère's circuital law**, discovered by André-Marie Ampère in 1826,[1] relates the integrated magnetic field around a closed loop to the electric current passing through the loop. James Clerk Maxwell derived it again using hydrodynamics in his 1861 paper On Physical Lines of Force and it is now one of the Maxwell equations, which form the basis of classical electromagnetism.

Integral form

In SI units (cgs units are later), the "integral form" of the original Ampère's circuital law is a line integral of the magnetic field around some closed curve C (arbitrary but must be closed). The curve C in turn bounds both a surface S which the electric current passes through (again arbitrary but not closed—since no three-dimensional volume is enclosed by S), and encloses the current. The mathematical statement of the law is a relation between the total amount of magnetic field around some path (line integral) due to the current which passes through that enclosed path (surface integral). It can be written in a number of forms.[2][3]

In terms of **total** current, which includes both free and **bound** current, the line integral of the magnetic B-field (in tesla, T) around closed curve C is proportional to the total current I_{enc} passing through a surface S (enclosed by C):

$$\oint_C \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 \iint_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} = \mu_0 I_{enc}$$

where $\mathbf{J}$ is the total current density (in ampere per square metre, Am^{-2}).

Alternatively in terms of **free** current, the line integral of the magnetic H-field (in ampere per metre, Am^{-1}) around closed curve C equals the free current $I_{f,enc}$ through a surface S:

$$\oint_C \mathbf{H} \cdot \mathrm{d}\boldsymbol{\ell} = \iint_S \mathbf{J}_f \cdot \mathrm{d}\mathbf{S} = I_{f,enc}$$

where $\mathbf{J}_f$ is the free current density only. Furthermore

-

$\oint_C$

- is the closed line integral around the closed curve C,
-

$\iint_S$

- denotes a 2d surface integral over S enclosed by C

- · is the vector dot product.
- dℓ is an infinitesimal element (a differential) of the curve C (i.e. a vector with magnitude equal to the length of the infinitesimal line element, and direction given by the tangent to the curve C)
- dS is the vector area of an infinitesimal element of surface S (that is, a vector with magnitude equal to the area of the infinitesimal surface element, and direction normal to surface S. The direction of the normal must correspond with the orientation of C by the right hand rule), see below for further explanation of the curve C and surface S.

The **B** and **H** fields are related by the constitutive equation

$$\mathbf{B} = \mu_0 \mathbf{H}$$

where μ0 is the magnetic constant.

There are a number of ambiguities in the above definitions that require clarification and a choice of convention.

1. First, three of these terms are associated with sign ambiguities: the line integral

$$\oint_C$$

1. could go around the loop in either direction (clockwise or counterclockwise); the vector area dS could point in either of the two directions normal to the surface; and Ienc is the net current passing through the surface S, meaning the current passing through in one direction, minus the current in the other direction—but either direction could be chosen as positive. These ambiguities are resolved by the right-hand rule: With the palm of the right-hand toward the area of integration, and the index-finger pointing along the direction of line-integration, the outstretched thumb points in the direction that must be chosen for the vector area dS. Also the current passing in the same direction as dS must be counted as positive. The right hand grip rule can also be used to determine the signs.
2. Second, there are infinitely many possible surfaces S that have the curve C as their border. (Imagine a soap film on a wire loop, which can be deformed by moving the wire). Which of those surfaces is to be chosen? If the loop does not lie in a single plane, for example, there is no one obvious choice. The answer is that it does not matter; it can be proven that any surface with boundary C can be chosen.

Differential form

By the Kelvin–Stokes theorem, this equation can also be written in a "differential form". Again, this equation only applies in the case where the electric field is constant in time, meaning the currents are steady (time-independent, else the magnetic field would change with time); see below for the more general form. In SI units, the equation states for total current:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

and for free current

$$\nabla \times \mathbf{H} = \mathbf{J}_f$$

where ∇× is the curl operator.

Note on free current versus bound current

The electric current that arises in the simplest textbook situations would be classified as "free current"—for example, the current that passes through a wire or battery. In contrast, "bound current" arises in the context of bulk materials that can be magnetized and/or polarized. (All materials can to some extent.)

When a material is magnetized (for example, by placing it in an external magnetic field), the electrons remain bound to their respective atoms, but behave as if they were orbiting the nucleus in a particular direction, creating a microscopic current. When the currents from all these atoms are put together, they create the same effect as a macroscopic current, circulating perpetually around the magnetized object. This magnetization current J_M is one contribution to "bound current".

The other source of bound current is bound charge. When an electric field is applied, the positive and negative bound charges can separate over atomic distances in polarizable materials, and when the bound charges move, the polarization changes, creating another contribution to the "bound current", the polarization current J_P.

The total current density J due to free and bound charges is then:

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_M + \mathbf{J}_P$$

with J_f the "free" or "conduction" current density.

All current is fundamentally the same, microscopically. Nevertheless, there are often practical reasons for wanting to treat bound current differently from free current. For example, the bound current usually originates over atomic dimensions, and one may wish to take advantage of a simpler theory intended for larger dimensions. The result is that the more microscopic Ampère's law, expressed in terms of **B** and the microscopic current (which includes free, magnetization and polarization currents), is sometimes put into the equivalent form below in terms of **H** and the free current only. For a detailed definition of free current and bound current, and the proof that the two formulations are equivalent, see the "proof" section below.

Maxwell's equations

Maxwell's equations are a set of partial differential equations that, together with the Lorentz force law, form the foundation of classical electrodynamics, classical optics, and electric circuits. These fields in turn underlie modern electrical and communications technologies. Maxwell's equations describe how electric and magnetic fields are generated and altered by each other and by charges and currents. They are named after the Scottish physicist and mathematician James Clerk Maxwell who published an early form of those equations between 1861 and 1862.

The equations have two major variants. The "microscopic" set of Maxwell's equations uses total charge and total current, including the complicated charges and currents in materials at the atomic scale; it has universal applicability, but may be unfeasible to calculate. The "macroscopic" set of Maxwell's equations defines two new auxiliary fields that describe large-scale behavior without having to consider these atomic scale details, but it requires the use of parameters characterizing the electromagnetic properties of the relevant materials.

The term "Maxwell's equations" is often used for other forms of Maxwell's equations. For example, space-time formulations are commonly used in high energy and gravitational physics. These formulations defined on space-time, rather than space and time separately are manifestly[1] compatible with special and general relativity. In quantum mechanics, versions of Maxwell's equations based on the electric and magnetic potentials are preferred.

Since the mid-20th century, it has been understood that Maxwell's equations are not exact laws of the universe, but are a classical approximation to the more accurate and fundamental theory of quantum electrodynamics. In most cases, though, quantum deviations from Maxwell's equations are immeasurably small. Exceptions occur when the particle nature of light is important or for very strong electric fields.

The precise formulation of the Maxwell equation depends on the precise definition of the quantities involved. Conventions differ with the unit systems because various definitions (and dimensions) are changed by absorbing dimensionful factors like the speed of light c. This makes constants come out differently. The equations in this section are given in the convention used with SI units. Other units commonly used are Gaussian units based on the cgs system,[4] Lorentz–Heaviside units (used mainly in particle physics), and Planck units (used in theoretical physics). See below for the formulation with Gaussian units.

The following equations are the conventional formulation of the Maxwell equations in terms of vector calculus using time dependent vector fields. Symbols in **bold** represent vector quantities, and symbols in *italics* represent scalar quantities. The definitions of terms used in the two tables of equations are given in another table immediately following. For a detailed description of the differences between the microscopic (total charge and current)[note 2] and macroscopic (free charge and current) variants of Maxwell's equations, see below.

Formulation	Name	"Microscopic" equation	"Macroscopic" equation
Integral	Gauss's law	$\oint_{\partial\Omega} \mathbf{E}\cdot d\mathbf{S} = \frac{Q(V)}{\varepsilon_0}$	$\oint_{\partial\Omega} \mathbf{D}\cdot d\mathbf{S} = Q_f(V)$
	Gauss's law for magnetism	$\oint_{\partial\Omega} \mathbf{B}\cdot d\mathbf{S} = 0$	Same as microscopic
	Maxwell–Faraday equation (Faraday's law of induction)	$\oint_{\partial\Sigma} \mathbf{E}\cdot d\boldsymbol{\ell} = -\iint_{\Sigma} \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S}$	Same as microscopic
	Ampère's circuital law (with Maxwell's correction)	$\oint_{\partial\Sigma} \mathbf{B}\cdot d\boldsymbol{\ell} = \mu_0 I + \mu_0\varepsilon_0 \iint_{\Sigma} \frac{\partial \mathbf{E}}{\partial t}\cdot d\mathbf{S}$	$\oint_{\partial\Sigma} \mathbf{H}\cdot d\boldsymbol{\ell} = I_f + \iint_{\Sigma} \frac{\partial \mathbf{D}}{\partial t}\cdot d\mathbf{S}$
Differential	Gauss's law	$\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}$	$\nabla\cdot\mathbf{D} = \rho_f$
	Gauss's law for magnetism	$\nabla\cdot\mathbf{B} = 0$	Same as microscopic
	Maxwell–Faraday equation (Faraday's law of induction)	$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	Same as microscopic
	Ampère's circuital law (with Maxwell's correction)	$\nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\varepsilon_0\frac{\partial \mathbf{E}}{\partial t}$	$\nabla\times\mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$

Vacuum equations, electromagnetic waves and speed of light

Further information: Electromagnetic wave equation and Sinusoidal plane-wave solutions of the electromagnetic wave equation

This 3D diagram shows a plane linearly polarized wave propagating from left to right with the same wave equations where $\mathbf{E} = E_0 \sin(-\omega t + \mathbf{k} \cdot \mathbf{r})$ and $\mathbf{B} = B_0 \sin(-\omega t + \mathbf{k} \cdot \mathbf{r})$

In a region with no charges ($\rho = 0$) and no currents ($J = 0$), such as in a vacuum, Maxwell's equations reduce to:

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}.$$

where

$$c = 1/\sqrt{\mu_0 \varepsilon_0}$$

= 2.99792458×108 m/s. Taking the curl (

$$\nabla \times$$

) of the curl equations, and using the identity

$$\nabla \times (\nabla \times) = \nabla (\nabla \cdot) - \nabla^2$$

we obtain the wave equations

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = 0, \quad \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = 0$$

which identify c with the speed of light in free space. In addition, **E** and **B** are mutually perpendicular to each other and the direction of wave propagation, and are in phase with each other. A sinusoidal plane wave is one special solution of these equations. Maxwell's equations explain how these waves can physically propagate through space. The changing magnetic field creates a changing electric field through Faraday's law. In turn, that electric field creates a changing magnetic field through Maxwell's correction to Ampère's law. This perpetual cycle allows these waves, now known as electromagnetic radiation, to move through space at velocity c.

Poynting's theorem

In electrodynamics, Poynting's theorem is a statement of energy conservation for the electromagnetic field, in the form of a partial differential equation, due to the British physicist John Henry Poynting.[1] Poynting's theorem is analogous to the work-energy theorem in classical mechanics, and mathematically similar to the continuity equation, because it relates the energy stored in the electromagnetic field to the work done on a charge distribution (i.e. an electrically charged object), through energy flux.

Statement

In words, the theorem is an energy balance:[2]

The rate of energy transfer (per unit volume) from a region of space equals the rate of work done on a charge distribution plus the energy flux leaving that region.

Mathematically, this is summarized in **differential form** as:

$$-\frac{\partial u}{\partial t} = \nabla \cdot \mathbf{S} + \mathbf{J} \cdot \mathbf{E}$$

where ∇•S is the divergence of the Poynting vector (energy flow) and J•E is the rate at which the fields do work on a charged object (J is the free current density corresponding to the motion of charge, E is the electric field, and • is the dot product). The energy density u is given by:[3]

$$u = \frac{1}{2}\left(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}\right).$$

in which **D** is the electric displacement field, **B** is the magnetic flux density and **H** the magnetic field strength. ε0 is the electric constant and μ0 is the magnetic constant. Since the charges are free to move, and the **D** and **H** fields bypass any bound charges and currents in the charge distribution (by their definition), J is the free current density, not the total.

Using the divergence theorem, Poynting's theorem can be rewritten in **integral form**:

$$-\frac{\partial}{\partial t}\int_V u \, dV = \oiint_{\partial V} \mathbf{S} \cdot d\mathbf{A} + \int_V \mathbf{J} \cdot \mathbf{E} \, dV$$

where

$$\partial V$$

is the boundary of a volume V. The shape of the volume is arbitrary but fixed for the calculation.

Derivation

Considering the statement in words above - there are three elements to the theorem, which involve writing energy transfer (per unit time) as volume integrals:[2]

1. Since u is the energy density, integrating over the volume of the region gives the total energy U stored in the region, then taking the (partial) time derivative gives the rate of change of energy:

$$U = \int_V u dV \ \rightarrow \ \frac{\partial U}{\partial t} = \frac{\partial}{\partial t}\int_V u dV = \int_V \frac{\partial u}{\partial t} dV.$$

1. The energy flux leaving the region is the surface integral of the Poynting vector, and using the divergence theorem this can be written as a volume integral:

$$\oiint_{\partial V} \mathbf{S} \cdot d\mathbf{A} = \int_V \nabla \cdot \mathbf{S} dV.$$

1. The Lorentz force density f on a charge distribution, integrated over the volume to get the total force F, is

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B} \ \rightarrow \ \int_V \mathbf{f} dV = \mathbf{F} = \int_V (\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}) dV,$$

where ρ is the charge density of the distribution and v its velocity. Since

$$\mathbf{J} = \rho\mathbf{v}$$

, the rate of work done by the force is

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \mathbf{F} \cdot \mathbf{v} = \int_V (\rho\mathbf{E} \cdot \mathbf{v} + \rho\mathbf{v} \times \mathbf{B} \cdot \mathbf{v}) dV \ \rightarrow \ \mathbf{F} \cdot \mathbf{v} = \int_V \mathbf{E} \cdot \mathbf{J} dV.$$

So by conservation of energy, the balance equation for the energy flow per unit time is the integral form of the theorem:

$$-\int_V \frac{\partial u}{\partial t} dV = \int_V \nabla \cdot \mathbf{S} dV + \int_V \mathbf{J} \cdot \mathbf{E} dV,$$

and since the volume V is arbitrary, this is true for all volumes, implying

$$-\frac{\partial u}{\partial t} = \nabla \cdot \mathbf{S} + \mathbf{J} \cdot \mathbf{E},$$

which is Poynting's theorem in differential form.

Poynting vector

Main article: Poynting vector

From the theorem, the actual form of the Poynting vector S can be found. The time derivative of the energy density (using the product rule for vector dot products) is

$$\frac{\partial u}{\partial t} = \frac{1}{2}\left(\mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t} + \mathbf{D}\cdot\frac{\partial \mathbf{E}}{\partial t} + \mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t} + \mathbf{B}\cdot\frac{\partial \mathbf{H}}{\partial t}\right) = \mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t} + \mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t},$$

using the constitutive relations

$$\mathbf{D} = \epsilon_0\mathbf{E}, \quad \mathbf{B} = \mu_0\mathbf{H}.$$

The partial time derivatives suggest using two of Maxwell's Equations. Taking the dot product of Faraday's Law with H:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E} \;\rightarrow\; \mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t} = -\mathbf{H}\cdot\nabla\times\mathbf{E},$$

next taking the dot product of the Ampère-Maxwell law equation with E:

$$\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} = \nabla\times\mathbf{H} \;\rightarrow\; \mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t} + \mathbf{E}\cdot\mathbf{J} = \mathbf{E}\cdot\nabla\times\mathbf{H}.$$

Collecting the results so far gives:

$$\begin{aligned}-\nabla\cdot\mathbf{S} &= \frac{\partial u}{\partial t} + \mathbf{J}\cdot\mathbf{E}\\ &= \left(\mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t} + \mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t}\right) + \mathbf{J}\cdot\mathbf{E}\\ &= \mathbf{E}\cdot\nabla\times\mathbf{H} - \mathbf{H}\cdot\nabla\times\mathbf{E},\end{aligned}$$

then, using the vector calculus identity:

$$\nabla \cdot \mathbf{E} \times \mathbf{H} = \mathbf{H} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H},$$

gives an expression for the Poynting vector:

$$\mathbf{S} = \mathbf{E} \times \mathbf{H},$$

which physically means the energy transfer due to time-varying electric and magnetic fields is perpendicular to the fields.

Vector potential

In vector calculus, a **vector potential** is a vector field whose curl is a given vector field. This is analogous to a scalar potential, which is a scalar field whose gradient is a given vector field.

Formally, given a vector field v, a *vector potential* is a vector field A such that

$$\mathbf{v} = \nabla \times \mathbf{A}.$$

If a vector field v admits a vector potential A, then from the equality

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

(divergence of the curl is zero) one obtains

$$\nabla \cdot \mathbf{v} = \nabla \cdot (\nabla \times \mathbf{A}) = 0,$$

which implies that v must be a solenoidal vector field.

Scalar potential

A **scalar potential** is a fundamental concept in vector analysis and physics (the adjective *scalar* is frequently omitted if there is no danger of confusion with vector potential). The scalar potential is an example of a scalar field. Given a vector field F, the scalar potential *P* is defined such that:

$$\mathbf{F} = -\nabla P = -\left(\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial P}{\partial z}\right)$$

.[1]

where ∇P is the gradient of P and the second part of the equation is minus the gradient for a function of the Cartesian coordinates x,y,z.[2] In some cases, mathematicians may use a positive sign in front of the gradient to define the potential.[3] Because of this definition of P in terms of the gradient, the direction of F at any point is the direction of the steepest decrease of P at that point, its magnitude is the rate of that decrease per unit length.

In order for F to be described in terms of a scalar potential only, the following have to be true:

1.

$$-\int_a^b \mathbf{F} \cdot d\mathbf{l} = P(\mathbf{b}) - P(\mathbf{a})$$

1. , where the integration is over a Jordan arc passing from location a to location b and P(b) is P evaluated at location b.

2.

$$\oint \mathbf{F} \cdot d\mathbf{l} = 0$$

1. , where the integral is over any simple closed path, otherwise known as a Jordan curve.
2.

$$\nabla \times \mathbf{F} = 0$$

1.

The first of these conditions represents the fundamental theorem of the gradient and is true for any vector field that is a gradient of a differentiable single valued scalar field P. The second condition is a requirement of F so that it can be expressed as the gradient of a scalar function. The third condition re-expresses the second condition in terms of the curl of F using the fundamental theorem of the curl. A vector field F that satisfies these conditions is said to be irrotational (Conservative).

Scalar potentials play a prominent role in many areas of physics and engineering. The gravity potential is the scalar potential associated with the gravity per unit mass, i.e., the acceleration due to the field, as a function of position. The gravity potential is the gravitational potential energy per unit mass. In electrostatics the electric potential is the scalar potential associated with the electric field, i.e., with the electrostatic force per unit charge. The electric potential is in this case the electrostatic potential energy per unit charge. In fluid dynamics, irrotational lamellar fields have a scalar potential only in the special case when it is a Laplacian field. Certain aspects of the nuclear force can be described by a Yukawa potential. The potential play a prominent role in the Lagrangian and Hamiltonian formulations of classical mechanics. Further, the scalar potential is the fundamental quantity in quantum mechanics.

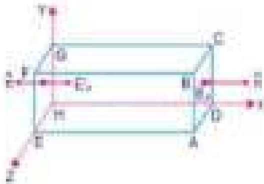
Not every vector field has a scalar potential. Those that do are called conservative, corresponding to the notion of conservative force in physics. Examples of non-conservative forces include frictional forces, magnetic forces, and in fluid mechanics a solenoidal field velocity field. By the Helmholtz decomposition theorem however, all vector fields can be describable in terms of a scalar potential and corresponding vector potential. In electrodynamics the electromagnetic scalar and vector potentials are known together as the electromagnetic four-potential.

Transverse Nature of Electromagnetic Waves

In an electromagnetic wave, the electric and magnetic field vectors oscillate perpendicular to the direction of propagation of electromagnetic wave. This shows that the electromagnetic waves are transverse in character.

This may be proved as follows :

Let a plane electromagnetic wave propagate along positive *X*-axis. Then the propagating wavefront will be in *Y*—*Z* plane. *ABCD* is a portion of wavefront at any time *t*. The electric and magnetic field vectors at time *t* will be zero to the right of *ABCD*. To the left of *ABCD*, they will depend on *x* and *t* but not on *Y* and *Z*. Since we are considering a plane wave.



Consider a closed surface *ABCDEFGH*. This surface does not enclose any charge, therefore by Gauss's theorem

$$\oint_{ABCDEFGH} \vec{E}.\vec{dS} = 0$$

$$\text{or} \quad \oint_{ABCD} \vec{E}.\vec{dS} + \oint_{EFGH} \vec{E}.\vec{dS} + \oint_{ABEF} \vec{E}.\vec{dS} + \oint_{DCGH} \vec{E}.\vec{dS} + \oint_{AEHD} \vec{E}.\vec{dS} + \oint_{BFGC} \vec{E}.\vec{dS} = 0 \ldots (1)$$

As electric field does not depend on *Y* and *Z*

$$\oint_{ABFE} \vec{E}.\vec{dS} = -\oint_{DCGH} \vec{E}.\vec{dS} \text{ and } \oint_{AEHD} \vec{E}.\vec{dS} = -\oint_{BFGC} \vec{E}.\vec{dS}$$

In view of above, equation (1) gives

$$\oint_{ABCD} \vec{E}.\vec{dS} + \oint_{EFGH} \vec{E}.\vec{dS} = 0$$

$$\Rightarrow \quad \oint E_x\hat{i}.(dy\,dz\hat{i}) + \oint E_x'\hat{i}.(-dy\,dz\hat{i}) = 0$$

$$\text{or} \quad E_x\,dy\,dz = Ex'\,dy\,dz$$

$$\text{or} \quad E_x = E_x'$$

i.e., component of electric field along the direction of propagation is constant. As a constant field cannot produce a wave, this implies that

$E_x = 0$.

In a similar manner it may be shown that the component of magnetic field along the direction of propagation of wave is zero, *i.e.*,

$$B_z = 0.$$

This shows that the electric and magnetic fields have no component along the direction of propagation. Thus, in an electromagnetic wave field vectors are perpendicular to the direction of propagation of wave, *i.e.*, electromagnetic waves are transverse in nature.